International Journal of Engineering Sciences &Research Technology

(A Peer Reviewed Online Journal)

Impact Factor: 5.164

IJESRT

ISSN: 2277-9655
Impact Factor: 5.164
CODEN: IJESS7

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

ON BINARY QUADRATIC EQUATION

$5x^2 - 6y^2 = 5$

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DOI: 10.5281/zenodo.2668940

ABSTRACT

Non-homogeneous binary quadratic equation representing hyperbola given by $5x^2 - 6y^2 = 5$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbola and parabola are presented. Also, employing the solutions of the given equation, is constructed.

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

1. INTRODUCTION

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a,b,c \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-18].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by, $5x^2 - 6y^2 = 5$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also, employing the solutions of the given equation, is constructed.

2. METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non- zero distinct integral solution is

$$5x^2 - 6y^2 = 5 \tag{1}$$

Introduce the linear transformation

$$x = X + 6T, y = X + 5T \tag{2}$$

From (1) & (2) we have

$$X^2 = 30T^2 - 5 \tag{3}$$

whose smallest positive integer solution is

$$X_0 = 5, T_0 = 1$$

To obtain the other solutions of (3), consider the pell equation ,

$$X^2 = 30T^2 + 1 \tag{4}$$

whose smallest positive integer solution is

$$\tilde{X}_0 = 11, \tilde{T}_0 = 2$$

The general solution of (4) is given by,

ISSN: 2277-9655
Impact Factor: 5.164
CODEN: IJESS7

$$\tilde{X}_n = \frac{1}{2} f_n, \tilde{T}_n = \frac{1}{2\sqrt{30}} g_n$$

where

$$f_n = [(11+2\sqrt{30})^{n+1} + (11-2\sqrt{30})^{n+1}]$$
$$g_n = [(11+2\sqrt{30})^{n+1} - (11-2\sqrt{30})^{n+1}]$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solutions of (1) are given by,

$$2\sqrt{30}x_{n+1} = 11\sqrt{30} f_n + 60 g_n$$
$$2\sqrt{30}y_{n+1} = 10\sqrt{30} f_n + 55 g_n$$

The recurrence relations satisfied by x and y are given by,

$$x_{n+1} - 22x_{n+2} + x_{n+3} = 0$$
$$y_{n+1} - 22y_{n+2} + y_{n+3} = 0$$

Some numerical examples of x and y satisfying (1) are given in the table 1 below.

Table 1: Numerical examples

n	x_n	y_n
0	11	10
1	241	220
2	5291	4830
3	116161	106040
4	2550251	2328050

From the above table, we observe some interesting relations among the solutions which are presented below:

- x_n values are always odd .
- y_n values are always even.

- **Each of the following expressions is a nasty number.**
  - $[264x_{2n+2} - 12x_{2n+3} + 12]$
  - $\frac{1}{11}[2898x_{2n+2} - 6x_{2n+4} + 132]$
  - $[132x_{2n+2} - 144y_{2n+2} + 12]$
  - $\frac{1}{11}[2892x_{2n+2} - 144y_{2n+3} + 132]$
  - $\frac{1}{241}[63492x_{2n+2} - 144y_{2n+4} + 2892]$
  - $[5796x_{2n+3} - 264x_{2n+4} + 12]$
  - $[12x_{2n+3} - 288y_{2n+2} + 12]$

- $[2892x_{2n+3} - 3168y_{2n+3} + 12]$
- $[5772x_{2n+3} - 288y_{2n+4} + 12]$
- $\frac{1}{241}[132x_{2n+4} - 69552y_{2n+2} + 2892]$
- $\frac{1}{11}[2892x_{2n+4} - 69552y_{2n+3} + 132]$
- $[63492x_{2n+4} - 69552y_{2n+4} + 12]$
- $\frac{1}{5}[66y_{2n+3} - 1446y_{2n+2} + 60]$
- $\frac{1}{10}[6y_{2n+4} - 2886y_{2n+2} + 120]$
- $\frac{1}{5}[1446y_{2n+4} - 31746y_{2n+3} + 60]$

➢ **Each of the following expressions is a cubical integer:**

- $[44x_{3n+3} - 2x_{3n+4} + 132x_{n+1} - 6x_{n+2}]$
- $\frac{1}{11}[483x_{3n+3} - x_{3n+5} + 1449x_{n+1} - 3x_{n+3}]$
- $[22x_{3n+3} - 24y_{3n+3} + 66x_{n+1} - 72y_{n+1}]$
- $\frac{1}{11}[482x_{3n+3} - 24y_{3n+4} + 1446x_{n+1} - 72y_{n+2}]$
- $\frac{1}{241}[10582x_{3n+3} - 24y_{3n+5} + 31746x_{n+1} - 72y_{n+3}]$
- $[966x_{3n+4} - 44x_{3n+5} + 2898x_{n+2} - 132x_{n+3}]$
- $[2x_{3n+4} - 48y_{3n+3} + 6x_{n+2} - 144y_{n+1}]$
- $[482x_{3n+4} - 528y_{3n+4} + 1446x_{n+2} - 1584y_{n+2}]$
- $[962x_{3n+4} - 48y_{3n+5} + 2886x_{n+2} - 144y_{n+3}]$
- $\frac{1}{241}[22x_{3n+5} - 11592y_{3n+3} + 66x_{n+3} - 34776y_{n+1}]$
- $\frac{1}{11}[482x_{3n+5} - 11592y_{3n+4} + 1446x_{n+3} - 34776y_{n+2}]$
- $[10582x_{3n+5} - 11592y_{3n+5} + 31746x_{n+3} - 34776y_{n+3}]$
- $\frac{1}{5}[11y_{3n+4} - 241y_{3n+3} + 33y_{n+2} - 723y_{n+1}]$
- $\frac{1}{10}[y_{3n+5} - 481y_{3n+3} + 3y_{n+3} - 1443y_{n+1}]$
- $\frac{1}{5}[241y_{3n+5} - 5291y_{3n+4} + 723y_{n+3} - 15873y_{n+2}]$

- **Each of the following expressions is a biquadratic integer:**
  - $[44x_{4n+4} - 2x_{4n+5} + 176x_{2n+2} - 8x_{2n+3} + 6]$
  - $\frac{1}{11}[483x_{4n+4} - x_{4n+6} + 1932x_{2n+2} - 4x_{2n+4} + 66]$
  - $[22x_{4n+4} - 24y_{4n+4} + 88x_{2n+2} - 96y_{2n+2} + 6]$
  - $\frac{1}{11}[482x_{4n+4} - 24y_{4n+5} + 1928x_{2n+2} - 96y_{2n+3} + 66]$
  - $\frac{1}{241}[10582x_{4n+4} - 24y_{4n+6} + 42328x_{2n+2} - 96y_{2n+4} + 1446]$
  - $[966x_{4n+5} - 44x_{4n+6} + 3864x_{2n+3} - 176x_{2n+4} + 6]$
  - $[2x_{4n+5} - 48y_{4n+4} + 8x_{2n+3} - 192y_{2n+2} + 6]$
  - $[482x_{4n+5} - 528y_{4n+5} + 1928x_{2n+3} - 2112y_{2n+3} + 6]$
  - $[962x_{4n+5} - 48y_{4n+6} + 3848x_{2n+3} - 192y_{2n+4} + 6]$
  - $\frac{1}{241}[22x_{4n+6} - 11592y_{4n+4} + 88x_{2n+4} - 46368y_{2n+2} + 1446]$
  - $\frac{1}{11}[482x_{4n+6} - 11592y_{4n+5} + 1928x_{2n+4} - 46368y_{2n+3} + 66]$
  - $[10582x_{4n+6} - 11592y_{4n+6} + 42328x_{2n+4} - 46368y_{2n+4} + 6]$
  - $\frac{1}{5}[11y_{4n+5} - 241y_{4n+4} + 44y_{2n+3} - 964y_{2n+2} + 30]$
  - $\frac{1}{10}[y_{4n+6} - 481y_{4n+4} + 4y_{2n+4} - 1924y_{2n+2} + 60]$
  - $\frac{1}{5}[241y_{4n+6} - 5291y_{4n+5} + 964y_{2n+4} - 21164y_{2n+3} + 30]$

- **Each of the following expressions is a quintic integer:**
  - $[44x_{5n+5} - 2x_{5n+6} + 220x_{3n+3} - 10x_{3n+4} + 440x_{n+1} - 20x_{n+2}]$
  - $\frac{1}{11}[483x_{5n+5} - x_{5n+7} + 2415x_{3n+3} - 5x_{3n+5} + 4830x_{n+1} - 10x_{n+3}]$
  - $[22x_{5n+5} - 24y_{5n+5} + 110x_{3n+3} - 120y_{3n+3} + 220x_{n+1} - 240y_{n+1}]$
  - $\frac{1}{11}[482x_{5n+5} - 24y_{5n+6} + 2410x_{3n+3} - 120y_{3n+4} + 4810x_{n+1} - 240y_{n+2}]$
  - $\frac{1}{241}[10582x_{5n+5} - 24y_{5n+7} + 52910x_{3n+3} - 120y_{3n+5} + 105820x_{n+1} - 240y_{n+3}]$
  - $[966x_{5n+6} - 44x_{5n+7} + 4830x_{3n+4} - 220x_{3n+5} + 9660x_{n+2} - 440x_{n+3}]$
  - $[2x_{5n+6} - 48y_{5n+5} + 10x_{3n+4} - 240y_{3n+3} + 20x_{n+2} - 480y_{n+1}]$
  - $[482x_{5n+6} - 528y_{5n+6} + 2410x_{3n+4} - 2640y_{3n+4} + 4820x_{n+2} - 5280y_{n+2}]$
  - $[962x_{5n+6} - 48y_{5n+7} + 4810x_{3n+4} - 240y_{3n+5} + 9620x_{n+2} - 480y_{n+3}]$

ISSN: 2277-9655
Impact Factor: 5.164
CODEN: IJESS7

- $\frac{1}{241}[22x_{5n+7}-11592y_{5n+5}+110x_{3n+5}-57960y_{3n+3}+220x_{n+3}-115920y_{n+1}]$
- $\frac{1}{11}[482x_{5n+7}-11592y_{5n+6}+2410x_{3n+5}-57960y_{3n+4}+4820x_{n+3}-115920y_{n+2}]$
- $[10582x_{5n+7}-11592y_{5n+7}+52910x_{3n+5}-57960y_{3n+5}+105820x_{n+3}-115920y_{n+3}]$
- $\frac{1}{5}[11y_{5n+6}-241y_{5n+5}+55y_{3n+4}-1205y_{3n+3}+110y_{n+2}-2410y_{n+1}]$
- $\frac{1}{10}[y_{5n+7}-481y_{5n+5}+5y_{3n+5}-2405y_{3n+3}+10y_{n+3}-4810y_{n+1}]$
- $\frac{1}{5}[241y_{5n+7}-5291y_{5n+6}+1205y_{3n+5}-26455y_{3n+4}+2410y_{n+3}-52910y_{n+2}]$

- **Relations satisfied by the solutions are as follows:**

- $x_{n+3}=22x_{n+2}-x_{n+1}$.
- $12y_{n+1}=x_{n+2}-11x_{n+1}$
- $12y_{n+2}=11x_{n+2}-x_{n+1}$
- $12y_{n+3}=241x_{n+2}-11x_{n+1}$
- $11x_{n+2}=x_{n+1}-x_{n+3}$
- $132y_{n+1}=x_{n+3}-241x_{n+1}$
- $12y_{n+2}=x_{n+3}-x_{n+1}$
- $132y_{n+3}=241x_{n+3}-x_{n+1}$
- $x_{n+3}=241x_{n+1}+264y_{n+1}$
- $y_{n+2}=10x_{n+1}+11y_{n+1}$
- $y_{n+3}=220x_{n+1}+241y_{n+1}$
- $x_{n+3}=x_{n+1}-24y_{n+2}$
- $11y_{n+3}=10x_{n+1}+241y_{n+2}$
- $241x_{n+3}=x_{n+1}-264y_{n+3}$
- $12y_{n+1}=11x_{n+3}-241x_{n+1}$
- $12y_{n+2}=x_{n+3}-11x_{n+2}$
- $12y_{n+3}=11x_{n+3}-x_{n+2}$
- $11x_{n+3}=241x_{n+2}+12y_{n+1}$
- $11y_{n+2}=10x_{n+2}+y_{n+1}$
- $y_{n+3}=20x_{n+2}+y_{n+1}$
- $y_{n+3}=10x_{n+2}+11y_{n+2}$
- $241y_{n+2}=10x_{n+3}+11y_{n+1}$
- $241y_{n+3}=220x_{n+3}+y_{n+1}$

ISSN: 2277-9655
Impact Factor: 5.164
CODEN: IJESS7

- $x_{n+1} = x_{n+3} - 24y_{n+1}$
- $11y_{n+3} = 10x_{n+3} + y_{n+2}$
- $x_{n+1} = 241x_{n+3} - 264y_{n+2}$
- $y_{n+3} = 22y_{n+2} - y_{n+1}$
- $20x_{n+1} = 284509y_{n+1} - 589y_{n+3}$
- $20x_{n+2} = 6259679y_{n+1} - 12959y_{n+3}$
- $20x_{n+3} = 137428429y_{n+1} - 284509y_{n+3}$
- $2y_{n+2} = 571433y_{n+1} - 1183y_{n+3}$

Remarkable Observations:

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table 2 below:

Table 2: Hyperbolas

S.No:	Hyperbola	(X, Y)
1)	$30X^2 - Y^2 = 120$	$[44x_{n+1} - 2x_{n+2}, 11x_{n+2} - 241x_{n+1}]$
2)	$120X^2 - 121Y^2 = 58080$	$[483x_{n+1} - x_{n+3}, x_{n+3} - 481x_{n+1}]$
3)	$5X^2 - 6Y^2 = 20$	$[22x_{n+1} - 24y_{n+1}, 22y_{n+1} - 20x_{n+1}]$
4)	$5X^2 - 726Y^2 = 2420$	$[482x_{n+1} - 24y_{n+2}, 2y_{n+2} - 40x_{n+1}]$
5)	$5X^2 - 6Y^2 = 1161620$	$[10582x_{n+1} - 24y_{n+3}, 22y_{n+3} - 9660x_{n+1}]$
6)	$30X^2 - Y^2 = 120$	$[966x_{n+2} - 44x_{n+3}, 241x_{n+3} - 5291x_{n+2}]$
7)	$605X^2 - 6Y^2 = 2420$	$[2x_{n+2} - 48y_{n+1}, 482y_{n+1} - 20x_{n+2}]$
8)	$5X^2 - 6Y^2 = 20$	$[482x_{n+2} - 528y_{n+2}, 482y_{n+2} - 440x_{n+2}]$
9)	$605X^2 - 6Y^2 = 2420$	$[962x_{n+2} - 48y_{n+3}, 482y_{n+3} - 9660x_{n+2}]$
10)	$5X^2 - 6Y^2 = 1161620$	$[22x_{n+3} - 11592y_{n+1}, 10582y_{n+1} - 20x_{n+3}]$
11)	$5X^2 - 6Y^2 = 2420$	$[482x_{n+3} - 11592y_{n+2}, 10582y_{n+2} - 440x_{n+3}]$

ISSN: 2277-9655
Impact Factor: 5.164
CODEN: IJESS7

12)	$5X^2 - 6Y^2 = 20$	$[10582x_{n+3} - 11592y_{n+3}, 10582y_{n+3} - 9660x_{n+3}]$
13)	$X^2 - 30Y^2 = 100$	$[11y_{n+2} - 241y_{n+1}, 44y_{n+1} - 2y_{n+2}]$
14)	$X^2 - 3000Y^2 = 400$	$[y_{n+3} - 481y_{n+1}, 483y_{n+1} - y_{n+3}]$
15)	$X^2 - 30Y^2 = 100$	$[241y_{n+3} - 5291y_{n+2}, 966y_{n+2} - 44y_{n+3}]$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table 3 below:

Table 3: Parabolas

S.NO:	Parabola	(X, Y)
1)	$30X - Y^2 = 60$	$[44x_{2n+2} - 2x_{2n+3}, 11x_{n+2} - 241x_{n+1}]$
2)	$120X - 11Y^2 = 2640$	$[483x_{2n+2} - x_{2n+4}, x_{n+3} - 481x_{n+1}]$
3)	$5X - 6Y^2 = 10$	$[22x_{2n+2} - 24y_{2n+2}, 22y_{n+1} - 20x_{n+1}]$
4)	$5X - 66Y^2 = 110$	$[482x_{2n+2} - 24y_{2n+3}, 2y_{n+2} - 40x_{n+1}]$
5)	$1205X - 6Y^2 = 580810$	$[10582x_{2n+2} - 24y_{2n+4}, 22y_{n+3} - 9660x_{n+1}]$
6)	$30X - Y^2 = 60$	$[966x_{2n+3} - 44x_{2n+4}, 241x_{n+3} - 5291x_{n+2}]$
7)	$605X - 6Y^2 = 1210$	$[2x_{2n+3} - 48y_{2n+2}, 482y_{n+1} - 20x_{n+2}]$
8)	$5X - 6Y^2 = 10$	$[482x_{2n+3} - 528y_{2n+3}, 482y_{n+2} - 440x_{n+2}]$
9)	$605X - 6Y^2 = 1210$	$[962x_{2n+3} - 48y_{2n+4}, 482y_{n+3} - 9660x_{n+2}]$
10)	$1205X - 6Y^2 = 580810$	$[22x_{2n+4} - 11592y_{2n+2}, 10582y_{n+1} - 20x_{n+3}]$
11)	$55X - 6Y^2 = 1210$	$[482x_{2n+4} - 11592y_{2n+3}, 10582y_{n+2} - 440x_{n+3}]$

12)	$5X-6Y^2=10$	$[10582x_{2n+4}-11592y_{2n+4},$ $10582y_{n+3}-9660x_{n+3}]$
13)	$X-6Y^2=10$	$[11y_{2n+3}-241y_{2n+2},$ $44y_{n+1}-2y_{n+2}]$
14)	$X-300Y^2=20$	$[y_{2n+4}-481y_{2n+2},$ $483y_{n+1}-y_{n+3}]$
15)	$X-6Y^2=10$	$[241y_{2n+4}-5291y_{2n+3},$ $966y_{n+2}-44y_{n+3}]$

3. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the Diophantine equation, represented by hyperbola is given by $5x^2-6y^2=5$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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ISSN: 2277-9655
Impact Factor: 5.164
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